Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference **Rules** for Propositional Logic
- Using Rules of Inference to **Build Arguments**
- **Rules** of Inference for Quantified Statements
- **Building Arguments** for Quantified Statements

Valid Arguments

1. Valid argument in **Propositional Logic**

   Inference Rules

2. Valid argument in **Predicate Logic**

   Inference rules for propositional logic plus additional inference rules to handle variables and quantifiers.

The **rules of inference** are the essential building block in the construction of valid arguments.

Arguments in Propositional Logic

- An ***argument*** in propositional logic is a sequence of propositions. All but the final proposition are called *premises*. The last statement is the *conclusion*.
- **Premises are true statements.**
- **The argument is valid if the premises imply the conclusion.**
- An ***argument form*** is an argument that is valid no matter what propositions are substituted into its propositional variables.
- If the premises are $p_1, p_2, \ldots, p_n$ and the conclusion is q then

  $(p_1 \land p_2 \land \ldots \land p_n) \to q$ is a tautology.
- Inference rules are all simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic: Modus Ponens

$$\boxed{\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}}$$

Corresponding Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Why premises imply the conclusion ?

Truth table

Example:

Let p be "It is snowing."

Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."

"It is snowing."

"Therefore , I will study discrete math."

Modus Tollens

$\neg p \vee q$

$\neg q$

$$\boxed{\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}}$$

Corresponding Tautology:

$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

Why premises imply the conclusion ?

Truth table

Example:
Let p be "it is snowing."
Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete math."

"Therefore , it is not snowing."

Hypothetical Syllogism

$$\boxed{\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}}$$

Corresponding Tautology:
$((p \to q) \land (q \to r)) \to (p \to r)$

Example:
Let p be "it snows."
Let q be "I will study discrete math."
Let r be "I will get an A."

"If it snows, then I will study discrete math."
"If I study discrete math, I will get an A."

"Therefore , If it snows, I will get an A."

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
$(\neg p \wedge (p \vee q)) \to q$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math or I will study English literature."
"I will not study discrete math."

"Therefore , I will study English literature."

Addition

$$\boxed{\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}}$$

Corresponding Tautology:

$p \rightarrow (p \vee q)$

Example:
Let p be "I will study discrete math."
Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore, I will study discrete math or I will visit Las Vegas."

Simplification

$$\boxed{\begin{array}{l} p \land q \\ \hline \therefore q \end{array}}$$

Corresponding Tautology:

$(p \land q) \rightarrow p$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math and English literature"

"Therefore, I will study discrete math."

Conjunction

$$\boxed{\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}}$$

Corresponding Tautology:
$((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:
Let p be "I will study discrete math."
Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore, I will study discrete math and I will study English literature."

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology:
$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:
Let p be "I will study discrete math."
Let r be "I will study English literature."
Let q be "I will study databases."

"I will not study discrete math or I will study English literature."
"I will study discrete math or I will study databases."

"Therefore, I will study databases or I will English literature."

Using the Rules of Inference to Build Valid Arguments

- A ***valid argument*** is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$
\begin{array}{l}
S_1 \\
S_2 \\
. \\
. \\
. \\
S_n \\
\\
C \qquad \therefore
\end{array}
$$

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \to q)$$

Show that q is a conclusion.

Solution:

Simplification

Step	**Reason**
1. $p \wedge (p \to q)$	Premise
2. p	~~Conjunction~~ using (1)
3. $p \to q$	~~Conjunction~~ using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

Example 2:

- With these hypotheses:

  "It is not sunny this afternoon and it is colder than yesterday."

  "We will go swimming only if it is sunny."

  "If we do not go swimming, then we will take a canoe trip."

  "If we take a canoe trip, then we will be home by sunset."
- Using the inference rules, construct a valid argument for the conclusion:

  "We will be home by sunset."

Solution:

1. Choose propositional variables:

   p : "It is sunny this afternoon." *r* : "We will go swimming." *t* : "We will be home by sunset."

   q : "It is colder than yesterday." *s* : "We will take a canoe trip."
2. Translation into propositional logic:

$$\text{Hypotheses: } \neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow s,\ s \rightarrow t$$
$$\text{Conclusion: } t$$

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Valid Arguments

3. Construct the Valid Argument

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \to s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \to t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

- **Valid arguments** for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
  - Rules of Inference for Propositional Logic
  - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Universal Instantiation (UI)

$$\boxed{\begin{array}{l} \forall x P(x) \\ \hline \therefore P(c) \end{array}}$$

Example:

Our domain consists of all dogs and Fido is a dog.

“All dogs are cuddly.”

“Therefore, Fido is cuddly.”

Universal Generalization (UG)

$$\begin{array}{l} P(c) \text{ for an arbitrary } c \\ \hline \therefore \forall x P(x) \end{array}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\begin{array}{l} \exists x P(x) \\ \hline \therefore P(c) \text{ for some element } c \end{array}$$

Example:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

Existential Generalization (EG)

$$\begin{array}{l} P(c) \text{ for some element } c \\ \hline \therefore \exists x P(x) \end{array}$$

Example:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ P(a), \text{where } a \text{ is a particular} \\ \quad \text{element in the domain} \\ \hline \therefore Q(a) \end{array}$$

This rule could be used in the Socrates example.

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

"Someone who passed the first exam has not read the book."

follows from the premises

"A student in this class has not read the book."

"Everyone in this class passed the first exam."

Solution: Let $C(x)$ denote "x is in this class," $B(x)$ denote "x has read the book," and $P(x)$ denote "x passed the first exam."

First we translate the premises and conclusion into symbolic form.

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

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Using Rules of Inference

Valid Argument:

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)